THE PRIMITIVE COHOMOLOGY LATTICE OF A COMPLETE INTERSECTION

ARNAUD BEAUVILLE

Abstract. We describe the primitive cohomology lattice of a smooth even-dimensional complete intersection in projective space.

Introduction

Let X be a smooth complete intersection of degree d and even dimension n in projective space. We describe in this note the lattice structure of the primitive cohomology $H^n(X,\mathbf{Z})_{\mathrm{o}}$. Excluding the cubic surface and the intersection of two quadrics, we find

$$H^n(X,\mathbf{Z})_{\mathrm{o}} = A_{d-1} \overset{\perp}{\oplus} k\,E_8(\pm 1) \overset{\perp}{\oplus} \ell\, U \quad \text{or} \quad \langle -d\rangle \overset{\perp}{\oplus} k'\,E_8(\pm 1) \overset{\perp}{\oplus} \ell'\, U$$

where the numbers k,ℓ,k',ℓ' and the sign attributed to E_8 depend on the multidegree and dimension of X – see Theorem 2.3 for a precise statement. The proof is an easy consequence of classical facts on unimodular lattices together with the Hirzebruch formula for the Hodge numbers of X.

We warn the reader that there are many ways to write an indefinite lattice as an orthogonal sum of indecomposable ones; for instance, when $8\,|\,d$, both decompositions above hold. Still it might be useful to have a (semi-) uniform expression for this lattice. Related results, with a different point of view, appear in [L-W].

1. Unimodular lattices

We will use the following standard notations for lattices: U denotes the hyperbolic plane, and $\langle d\rangle$ the lattice $\mathbf{Z}\,e$ with $e^2 = d$. If L is a lattice, $L(-1)$ denotes the $\mathbf{Z}$-module L with the form $x \mapsto -x^2$; if n is a negative number, we put $n\,L := |n|\,L(-1)$.

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Let L be an odd unimodular lattice. A primitive vector $h \in L$ is said to be *characteristic* if $h \cdot x \equiv x^2 \pmod{2}$ for all $x \in L$; this is equivalent to saying that the orthogonal lattice $h^\perp$ is even ([L-W], Lemma 3.3).

Proposition 1.1. *Let L be a unimodular lattice, of signature (b^+, b^-), with $b^+, b^- \geq 2$; put $s := b^+ - b^-$, $t = \min(b^+, b^-)$, $u = \min(b^-, b^+ - d)$. Let h be a primitive vector in L of square $d > 0$, such that $h^\perp$ is even.*

1) *If L is even or $8 \,|\, d$ we have $h^\perp = \langle -d \rangle \overset{\perp}{\oplus} \frac{s}{8} E_8 \overset{\perp}{\oplus} (t-1)\, U$.*

2) *If L is odd and $d \leq b^+$, we have $h^\perp = A_{d-1} \overset{\perp}{\oplus} \frac{s-d}{8} E_8 \overset{\perp}{\oplus} u\, U$.*

Proof : A classical result of Wall [W] tells us that h is equivalent under $O(L)$ to any primitive vector v of square d, provided v is characteristic if so is h. If L is even, we choose a hyperbolic plane $U \subset L$ with a hyperbolic basis (e, f), and we put $v = e + \frac{d}{2} f$; then $v^\perp = \mathbf{Z}(e - \frac{d}{2} f) \overset{\perp}{\oplus} U^\perp$, and $U^\perp$ is an indefinite unimodular lattice, hence of the form $p\, E_8(\pm 1) \overset{\perp}{\oplus} q\, U$. Computing b^+ and b^- we find the above expressions for p and q.

Consider now the case when L is odd. We first observe that since h is characteristic, we have $d = h^2 \equiv s \pmod{8}$ ([S], V, Th. 2). Let

$$L' := (\overset{\perp}{\bigoplus_{i \leq d}} \mathbf{Z}\, e_i) \overset{\perp}{\oplus} \frac{s-d}{8} E_8 \overset{\perp}{\oplus} u\, U \quad \text{with} \quad e_1^2 = \ldots = e_d^2 = 1 \ .$$

L' is odd, indefinite and has the same signature as L, hence is isometric to L. We put $v = e_1 + \ldots + e_d$. The orthogonal of v in $\overset{\perp}{\oplus} \mathbf{Z}\, e_i$ is the root lattice A_{d-1}. By Wall's theorem $h^\perp$ is isometric to $v^\perp = A_{d-1} \overset{\perp}{\oplus} \frac{s-d}{8} E_8 \overset{\perp}{\oplus} u\, U$.

Suppose moreover that 8 divides d, so that $8 \,|\, s$. Then L is isomorphic to $\mathbf{Z}\, e \overset{\perp}{\oplus} \mathbf{Z}\, f \overset{\perp}{\oplus} \frac{s}{8} E_8 \overset{\perp}{\oplus} (t-1)\, U$, with $e^2 = 1$, $f^2 = -1$. Taking $v = (\frac{d}{4} + 1)e + (\frac{d}{4} - 1)f$ gives the result. □

2. Complete intersections

We will check that the hypotheses of the Proposition hold for the cohomology of complete intersections; the only non trivial point is the inequality $d \leq b^+$.

We will use the notations of [D1]. Let $\mathbf{d}=(d_1,\ldots,d_c)$ be a sequence of positive integers. We denote by $V_n(\mathbf{d})$ a smooth complete intersection of multidegree $\mathbf{d}$ in $\mathbf{P}^{n+c}$. We put

$$h^{p,q}(\mathbf{d})=\dim H^{p,q}(V_{p+q}(\mathbf{d})) \qquad \text{and} \qquad h^{p,q}_{\mathrm{o}}(\mathbf{d})=h^{p,q}(\mathbf{d})-\delta_{p,q}\ .$$

Lemma 2.1. $h^{p+1,q+1}(\mathbf{d})\geq h^{p,q}(\mathbf{d})$.

Proof : Following [D1] we introduce the formal generating series

$$H(\mathbf{d})=\sum_{p,q\geq 0} h^{p,q}_{\mathrm{o}}(\mathbf{d})y^pz^q \ \in \mathbf{Z}[[y,z]]\ ;$$

we define a partial order on $\mathbf{Z}[[y,z]]$ by writing $P\geq Q$ if $P-Q$ has non-negative coefficients. The assertion of the lemma is equivalent to $H(\mathbf{d})\geq yzH(\mathbf{d})$. The set $\mathcal{P}$ of formal series in $\mathbf{Z}[[y,z]]$ with this property is stable under addition and multiplication by any $P\geq 0$ in $\mathbf{Z}[[y,z]]$. The formula

$$H(d_1,\ldots,d_c)=\sum_{\substack{P\subset[1,d]\\ P\neq\varnothing}}\big[(1+y)(1+z)\big]^{|P|-1}\prod_{i\in P}H(d_i)$$

([D1], Cor. 2.4 (ii)) shows that it is enough to prove that $H(d)$ is in $\mathcal{P}$.

By [D1], Cor. 2.4 (i), we have $H(d)=\dfrac{P}{1-Q}$ with

$$P(y,z)=\sum_{i,j\geq 0}\binom{d-1}{i+j+1}y^iz^j \quad \text{and} \quad Q(y,z)=\sum_{i,j\geq 1}\binom{d}{i+j}y^iz^j\ .$$

Since $Q\geq yz$, we get $\dfrac{1-yz}{1-Q}=1+\dfrac{Q-yz}{1-Q}\geq 0$, hence $(1-yz)H\geq 0$. $\square$

Lemma 2.2. *Let $d=d_1\ldots d_c$. We have:*

a) $h^{p,p}(\mathbf{d})\geq d$;

b) $2h^{p+1,p-1}(\mathbf{d})+1\geq d$, *except in the following cases:*

- $\mathbf{d}=(2),(2,2)$;
- $p=1$, $\mathbf{d}=(3),(2,3),(2,2,2),(2,2,2,2)$;
- $p=2$, $\mathbf{d}=(2,2,2)$.

Proof : We first prove b) in the case $p=1$. Then $V_2(\mathbf{d})$ is a surface S. The canonical bundle K_S is $\mathcal{O}_S(e)$, with $e:=d_1+\ldots d_c-c-3$; therefore $K_S^2=e^2d$. The case $e\leq 0$ is immediate, so we assume $e\geq 1$.

Then the index $K_S^2 - 8\chi(\mathcal{O}_S)$ of the intersection form is negative [P]; if $e \geq 2$ we get $\chi(\mathcal{O}_S) > \frac{d}{2}$, hence $2h^{2,0}(\mathbf{d}) + 1 \geq d$.

If $e = 1$, we have $K_S = \mathcal{O}_S(1)$ hence $p_g = c+3$. The possibilities for $\mathbf{d}$ are $(5), (2,4), (3,3), (2,2,3)$ and $(2,2,2,2)$, and we have $2(c+3)+1 \geq d$ in each case except the last one. This also holds for $\mathbf{d} = (4)$, and the other cases are excluded.

Since the index is negative, we have $h^{1,1}(\mathbf{d}) > 2h^{2,0}(\mathbf{d}) + 1$; this implies that a) holds (for $p = 1$) except perhaps for $\mathbf{d} = (3), (2,2), (2,3), (2,2,2)$. But the corresponding $h^{1,1}$ is $7, 6, 19, 19$, which is always $> d$.

Now assume $p \geq 2$. a) follows from the previous case and Lemma 2.1; similarly it suffices to check b) for the values of $\mathbf{d}$ excluded in the case $p = 1$. Using the above formulas we find

$$h^{3,1}(3) = 1\ ,\ h^{3,1}(2,3) = 8\ ,\ h^{3,1}(2,2,2,2) = 27\ ,\ h^{4,2}(2,2,2) = 6\ ,$$

so that $2h^{p+1,p-1}(\mathbf{d}) + 1 \geq d$ for $p \geq 2$ in the three first cases and for $p \geq 3$ in the last one. $\square$

Theorem 2.3. *Let X be a smooth even-dimensional complete intersection in $\mathbf{P}^{n+c}$, of multidegree $\mathbf{d} = (d_1, \ldots, d_c)$. Let $d := d_1 \ldots d_c$ be the degree of X, and let e be the number of integers d_i which are even.*

Let (b^+, b^-) be the signature of the intersection form on $H^n(X, \mathbf{Z})$; we put

$$s = b^+ - b^- \quad , \quad t = \min(b^+, b^-) \quad , \quad u = \min(b^+ - d, b^-)\ .$$

We assume $\mathbf{d} \neq (2,2)$, and $\mathbf{d} \neq (3), (2,2,2,2)$ when $n = 2$. Then:

- $H^n(X, \mathbf{Z})_{\mathrm{o}} = \langle -d \rangle \overset{\perp}{\oplus} \frac{s}{8} E_8 \overset{\perp}{\oplus} (t-1)\, U$ *if $\binom{\frac{n}{2}+e}{e}$ is even;*
- $H^n(X, \mathbf{Z})_{\mathrm{o}} = A_{d-1} \overset{\perp}{\oplus} \frac{s-d}{8} E_8 \overset{\perp}{\oplus} u\, U$ *if $\binom{\frac{n}{2}+e}{e}$ is odd.*

For a hypersurface, for instance, we find a lattice of the form $A_{d-1} \overset{\perp}{\oplus} p\, E_8 \overset{\perp}{\oplus} q\, U$ if and only if d is odd, or d is even and n is divisible by 4.

Proof : We apply Proposition 1.1 with $L = H^n(X, \mathbf{Z})$. We take for h the class of a linear section of codimension $\frac{n}{2}$, so that $h^2 = d$.

By [L-W], Thm. 2.1 and Cor. 2.2, we know that

- h is primitive;
- $h^\perp$ is even;
- L is even or odd according to the parity of $\binom{\frac{n}{2}+e}{e}$.

To apply the Proposition we only need the inequalities $b^+ \geq d$ and $b^- \geq 2$. Note that the statement of the theorem holds trivially for $\mathbf{d} = (2)$, so we may assume $d \geq 3$. Let us write $n = 4k + 2\varepsilon$, with $\varepsilon \in \{0, 1\}$. By Hodge theory we have

$$b^+ = \sum_{\substack{p+q=n \\ p \text{ even}}} h^{p,q} + \varepsilon \qquad b^- = \sum_{\substack{p+q=n \\ p \text{ odd}}} h^{p,q} - \varepsilon \;;$$

when the inequalities a) and b) of Lemma 2.2 hold this implies $b^+ \geq d$ and $b^- \geq 2$, so Proposition 1.1 gives the result.

In the exceptional cases of Lemma 2.2 b), the lattice L is even and we have $b^+, b^- \geq 2$, so Proposition 1.1 still applies. □

Remark 2.4. The two first exceptions mentioned in the theorem are well-known ([D2], Prop. 5.2): we have $H^2(X, \mathbf{Z})_{\mathrm{o}} = E_6$ for a cubic surface, and $H^n(X, \mathbf{Z})_{\mathrm{o}} = D_{n+3}$ for a n-dimensional intersection of two quadrics. For an intersection of 4 quadrics in $\mathbf{P}^6$, we have $d = 16$, hence by Proposition 1.1

$$H^2(X, \mathbf{Z})_{\mathrm{o}} = \langle -16 \rangle \overset{\perp}{\oplus} 6\, E_8(-1) \overset{\perp}{\oplus} 15\, U \;.$$

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Laboratoire J.-A. Dieudonné, UMR 6621 du CNRS, Université de Nice, Parc Valrose, F-06108 Nice cedex 2, France

E-mail address: arnaud.beauville@unice.fr